

ON THE PROBLEM OF SEMIINFINITE BEAM OSCILLATION WITH INTERNAL DAMPING*

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ABSTRACT. We study the Cauchy problem for the equation of the form

$$(*) \quad \ddot{u}(t) + (\alpha A + B)\dot{u}(t) + (A + G)u(t) = 0$$

where A , B , and G are operators in a Hilbert space \mathcal{H} with A selfadjoint, $\sigma(A) = [0, \infty)$, $B \geq 0$ bounded, and G symmetric and A -subordinate in a certain sense. Spectral properties of the corresponding operator pencil $L(\lambda) := \lambda^2 I + \lambda(\alpha A + B) + A + G$ are studied, and existence and uniqueness of generalized and classical solutions of the Cauchy problem are proved. Equations of the type $(*)$ include, e.g., an abstract model for the problem of semiinfinite beam oscillations with internal damping.

This article is based on the lecture delivered at VII Crimean Autumn Mathematical School-Symposium on Spectral and Evolutionary Problems, Crimea, Ukraine, September 18–29, 1996.

INTRODUCTION

The aim of the present article is to study some class of differential equations and corresponding operator pencils in a Hilbert space, which provide abstract models for many problems in elasticity theory, hydrodynamics, control theory etc.

Consider, for example, a visco-elastic semiinfinite beam placed in viscous external medium. Its small transverse oscillations are described in dimensionless coordinates by the equation (cf. [P1])

$$(1) \quad \alpha \frac{\partial^5 u}{\partial t \partial x^4} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial}{\partial x} \left(g(x) \frac{\partial u}{\partial x} \right) + \beta(x) \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = 0, \quad x \geq 0, \quad t \geq 0.$$

Here $u(x, t)$ is the transverse displacement of the beam at point x and time t ; $\alpha > 0$ is a small parameter specifying internal damping, $\beta(x)$ determines external damping, and $g(x)$ describes tension force distribution.

Suppose for simplicity that the left beam end is clamped, i.e. that $u(x, t)$ satisfies the boundary conditions

$$(2) \quad u(0, t) = \frac{\partial u(x, t)}{\partial x} \Big|_{x=0} = 0,$$

*This work was partially supported by Russian Fund of Basic Research, grant No. 96-01-01292.
1991 *Mathematics Subject Classification.* 47A56, 47N20, 35P05.

Key words and phrases. Abstract Cauchy problems, operator pencils, spectral theory.

and let at the moment $t = 0$ the profile and velocity of the beam are

$$(3) \quad u(x, 0) = \psi_0(x), \quad \frac{\partial u(x, t)}{\partial t} \Big|_{t=0} = \psi_1(x).$$

We will represent the initial boundary-value problem (1)–(3) as a Cauchy problem for an abstract equation in the Hilbert space $L_2(0, \infty)$, and then will study the latter by means of unbounded operator theory and operator pencil theory. (See also [P1] and [H1] for some related results.)

1. ABSTRACT DIFFERENTIAL EQUATION AND OPERATOR PENCIL

Problem (1)–(3) can be written in the form of

$$(4) \quad \ddot{u}(t) + (\alpha A + B)\dot{u}(t) + (A + G)u(t) = 0,$$

$$(5) \quad u(0) = \psi_0, \quad \dot{u}(0) = \psi_1,$$

where $u(t)$ is a function taking its values into the Hilbert space $\mathcal{H} := L_2(0, \infty)$, $\psi_0, \psi_1 \in \mathcal{H}$, and A, B , and G are linear operators in \mathcal{H} defined by the equalities¹

$$(6) \quad \begin{aligned} (Ay)(x) &= y^{\text{iv}}(x), & \mathfrak{D}(A) &= \{y \in W_2^4(0, \infty) \mid y(0) = y'(0) = 0\}, \\ (By)(x) &= \beta(x)y(x), & \mathfrak{D}(B) &= H, \\ (Gy)(x) &= (g(x)y'(x))', & \mathfrak{D}(G) &= \mathfrak{D}(A). \end{aligned}$$

Suppose that the functions $g(x)$, $g'(x)$, and $\beta(x)$ are real, measurable, and essentially bounded; moreover, $g(x)$ and $\beta(x)$ belong to “the class \mathcal{K} ” ([B]), i. e. there exists a number $a > 0$ such that

$$\lim_{x \rightarrow \infty} \int_{x-a}^{x+a} |g(s)| ds = 0, \quad \lim_{x \rightarrow \infty} \int_{x-a}^{x+a} |\beta(s)| ds = 0.$$

Then the operators A, B and G possess the following properties:

- (H1) $A = A^* > 0$, the essential spectrum of the operator A coincides with the semiaxis $[0, \infty)$;
- (H2) $B = B^* \geq 0$ is bounded and $(A + I)$ -compact in the sense of quadratic forms (cf. [B], [RS]);
- (H3) G is symmetric, $A^{1/2}$ -subordinated, i.e. $|G| \leq g_0 A^{1/2} + g_1 I$ for some $g_0, g_1 > 0$, and $(A + I)$ -compact in the sense of quadratic forms.

In the sequel we will study abstract Cauchy problem (4)–(5) in the Hilbert space \mathcal{H} under hypotheses (H1)–(H3) on the operators A, B and G only, and will not exploit their concrete form (6) until section 4.

Let there exists a solution to equation (4) of the form $u(t) = e^{\lambda t}y$ with $\lambda \in \mathbb{C}$ and $y \in \mathcal{H}$. In order to find all such λ and y we get the spectral problem

$$[\lambda^2 I + \lambda(\alpha A + B) + A + G]y = 0$$

for the quadratic operator pencil

$$L(\lambda) := \lambda^2 I + \lambda(\alpha A + B) + A + G$$

in the Hilbert space \mathcal{H} .

¹ $\mathfrak{D}(T)$ denotes the domain of the operator T .

2. SPECTRAL PROPERTIES OF OPERATOR PENCIL $L(\lambda)$

Behavior of solutions to equation (4) depends heavily on the structure and localization of operator pencil $L(\lambda)$ spectrum, and so in this section we will briefly discuss some spectral properties of $L(\lambda)$.

First recall that the *spectrum* $\sigma(L)$ of the pencil $L(\lambda)$ is the complement in the complex plane \mathbb{C} to the set $\rho(L)$ of all *regular* points; here $\lambda_0 \in \rho(L)$ iff the operator $L(\lambda_0)$ is boundedly invertible and the inverse operator $L^{-1}(\lambda_0)$ is defined on the whole space \mathcal{H} . We distinguish in $\sigma(L)$ the *point spectrum*

$$\sigma_p(L) := \{\lambda_0 \in \sigma(L) \mid \text{Ker } L(\lambda_0) \neq \{0\}\};$$

and the *essential spectrum*

$$\sigma_{\text{ess}}(L) := \{\lambda_0 \in \sigma(L) \mid \text{the operator } L(\lambda_0) \text{ is not a Fredholm one}\}.$$

Any number $\lambda_0 \in \sigma_p(L)$ is called an *eigenvalue* (EV), and any nonzero vector $y_0 \in \text{Ker } L(\lambda_0)$ is called a corresponding *eigenvector* of the pencil $L(\lambda)$.

2.1. Essential spectrum. Let O denote the circle of radius $1/\alpha$ with centrum at the point $-1/\alpha$ and J denote the interval $(-\infty, -1/\alpha]$.

Theorem 1 ([H1]). *The essential spectrum $\sigma_{\text{ess}}(L)$ of the pencil $L(\lambda)$ coincides with the set $O \cup J$.*

2.2. Nonreal eigenvalues. Let $\Pi_- := \{z \in \mathbb{C} \mid \text{Re } z < 0\}$ be the left half-plane and numbers² b_+ and b_- (g_+ and g_-) denote the upper and lower bounds of the operator B (of the operator G , respectively). Note that due to hypotheses (H1)–(H3) we have the inequalities $\pm b_{\pm} \geq 0$ and $\pm g_{\pm} \geq 0$; moreover, $b_- = 0$.

Lemma 2. *All the nonreal EV's of the pencil $L(\lambda)$ belong to the set $\Pi_- \cap M \cap R$, where M is the set³*

$$M := \{\lambda \in \mathbb{C} \mid |\lambda + 1/\alpha|^2 \leq 1/\alpha^2 + g_1 + \sqrt{-2g_0 \text{Re } \lambda/\alpha}\}$$

and R is the ring⁴

$$R := \{\lambda \in \mathbb{C} \mid r_- \leq |\lambda - 1/\alpha|^2 \leq r_+\}$$

with the numbers r_{\pm} determined via the operators A , B , and G . In particular, if the operator G is bounded above (below), then we can put $r_+ := \frac{1}{\alpha} \sqrt{1 + \alpha^2 g_+}$ (respectively, $r_- := \frac{1}{\alpha} \sqrt{1 - \alpha b_+ + \alpha^2 g_-}$).

Proof. The assertion about Π_- and R (as well as the choice of r_{\pm}) was proved in [H2]. Next, it was shown in [H1] (cf. also [P1]) that for any $\gamma > 1/\alpha$ all the nonreal EV's are contained in the set $M_{\gamma} := \{\lambda \in \mathbb{C} \mid |\lambda + \gamma|^2 \leq \gamma^2 - g_1 + g_0^2/4(\gamma\alpha - 1)\}$, and the intersection $\bigcap_{\gamma > 1/\alpha} M_{\gamma}$ is easily seen to coincide with the set M .

²Some of the numbers b_{\pm} and g_{\pm} may equal $\pm\infty$.

³The numbers g_0 and g_1 were introduced in hypothesis (H3).

⁴Which can degenerate into a disc or a point.

2.3. The spectrum in the right half-plane.

Lemma 3. *The nonzero spectrum of the pencil $L(\lambda)$ in the closed right half-plane consists of the real isolated EV's; their number $\varkappa_1(L)$ counted according to multiplicities equals⁵ $\nu(A+G)$, the total multiplicity of negative spectrum of the operator $A+G$.*

Proof. It is a corollary of proposition 6 in [LSY]. Similar result was also proved in [LS] and [P2].

Corollary 4. *The point $\lambda = 0$ is an accumulation point of real EV's of the pencil $L(\lambda)$ from the right iff $\nu(A+G) = \infty$.*

Note that for concrete differential operators (6) the quantity $\nu(A+G)$ can be easily estimated from above, see section 4.

2.4. Accumulation of real EV's at the points $-1/\alpha$ and 0. Let for $k > -1/\alpha$ a number $\nu(k)$ denote the total multiplicity of negative spectrum of the operator $L(k)$.

Lemma 5 ([Hr1]). *Suppose that $\nu(-1/\alpha) = \infty$ ($\nu(0) = \infty$); then EV's from the interval $(-1/\alpha, 0)$ accumulate at the point $-1/\alpha$ (at the point 0, respectively).*

3. THE CAUCHY PROBLEM

3.1. Classical and generalized solutions. We start with the following definition.

Definition. Let S and T be closed operators in \mathcal{H} . A function $u(t) \in C^2(\mathbb{R}_+, \mathcal{H})$ is said to be a *classical solution* to the equation

$$(7) \quad \ddot{u}(t) + S\dot{u}(t) + Tu(t) = 0$$

if for any $t > 0$ we have $u(t) \in \mathfrak{D}(T)$, $\dot{u}(t) \in \mathfrak{D}(S)$, and equality (7) holds.

Fix a number $k_0 > \sup_{\lambda \in \sigma(L)} \operatorname{Re} \lambda$ and consider the pencil

$$\tilde{L}(\xi) := L(\xi + k_0) = \xi^2 I + \xi \tilde{B} + \tilde{C},$$

where $\tilde{B} := 2k_0 I + \alpha A + B \gg 0$ and $\tilde{C} := L(k_0) \gg 0$. It is easily seen that a function $u(t)$ is a classical solution to equation (4) iff the function $v(t) := e^{-k_0 t} u(t)$ is a classical solution to the equation

$$(8) \quad \tilde{L}\left(\frac{d}{dt}\right)v(t) := \ddot{v}(t) + \tilde{B}\dot{v}(t) + \tilde{C}v(t) = 0.$$

If in addition equalities (5) hold, then $v(t)$ satisfies the initial conditions

$$(9) \quad v(0) = \psi_0, \quad \dot{v}(0) = -k_0 \psi_0 + \psi_1.$$

⁵Therefore, the *instability index* $\varkappa(L)$ of the pencil $L(\lambda)$, i. e. the number of linearly independent increasing solutions to equation (4), is not less than $\varkappa_1(L)$. If $\lambda = 0$ is a singular critical point ([La]), then $\varkappa(L) > \varkappa_1(L)$.

Problem (8)–(9) now can be reduced to the first order system

$$(10) \quad \dot{\mathbf{V}}(t) = \tilde{\mathbb{T}} \mathbf{V}(t),$$

$$(11) \quad \mathbf{V}(0) = \boldsymbol{\psi} := \begin{pmatrix} \psi_0 \\ -k_0\psi_0 + \psi_1 \end{pmatrix},$$

in the space $\mathcal{H} \times \mathcal{H}$, where

$$\mathbf{V}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \quad \text{and} \quad \tilde{\mathbb{T}} = \begin{pmatrix} 0 & I \\ -\tilde{C} & -\tilde{B} \end{pmatrix}.$$

Actually it is more natural to consider system (10)–(11) not in the space $\mathcal{H} \times \mathcal{H}$, but in the so-called “energy” space $\mathbb{H} = \mathcal{H}_{1/2} \times \mathcal{H}$, where the Hilbert space scale \mathcal{H}_θ is generated by the operator \tilde{C} (namely, \mathcal{H}_θ coincides with $\mathfrak{D}(\tilde{C}^\theta)$ and is equipped with the norm $\|\phi\|_\theta := \|\tilde{C}^\theta \phi\|$, see for details [LM]). Then the operator $\tilde{\mathbb{T}}$ is closed and densely defined on the domain

$$\mathfrak{D}(\tilde{\mathbb{T}}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}_{1/2} \times \mathcal{H}_{1/2} \mid \tilde{C}x_1 + \tilde{B}x_2 \in \mathcal{H}_0 \right\}.$$

Now we define a solution to equation (10) to be any function $\mathbf{V}(t) \in C^1(\mathbb{R}_+, \mathbb{H})$ such that $\mathbf{V}(t) \in \mathfrak{D}(\tilde{\mathbb{T}})$ for all $t > 0$ and equality (10) is fulfilled.

It is easily seen that any classical solution $v(t)$ to equation (8) generates the solution $\mathbf{V}(t) := (v(t), \dot{v}(t))$ to equation (10). On the contrary, if $\mathbf{V}(t) = (v_1(t), v_2(t))$ is a solution to equation (10), then the function $v_1(t)$, which formally satisfies (8), may not be a classical solution to (8). Therefore it is natural to call the function $v_1(t)$ a *generalized solution* to equation (8).

3.2. Analyticity of the semigroup generated by the operator $\tilde{\mathbb{T}}$. First we will deal with generalized solution to problem (8)–(9). The solvability of corresponding system (10)–(11) depend essentially on the properties of the operator $\tilde{\mathbb{T}}$.

Theorem 6. *The operator $\tilde{\mathbb{T}}$ generates an analytic C_0 -semigroup of contractions \mathbb{U}_t in the space \mathbb{H} .*

Proof. According to [K], it suffices to prove that for some constant $C > 0$ and all $\xi \in \mathbb{C}$ with $\operatorname{Re} \xi > 0$ the inequality

$$(12) \quad \left\| \left(\tilde{\mathbb{T}} - \xi \mathbb{I} \right)^{-1} \right\|_{\mathfrak{B}(\mathbb{H})} \leq C/|\xi|$$

holds. The straightforward calculations show that the relation

$$\left(\tilde{\mathbb{T}} - \xi \mathbb{I} \right) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

implies

$$(13) \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left(\tilde{\mathbb{T}} - \xi \mathbb{I} \right)^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -\tilde{L}^{-1}(\xi)(\tilde{B} + \xi I) & -\tilde{L}^{-1}(\xi) \\ \tilde{L}^{-1}(\xi)\tilde{C} & -\xi\tilde{L}^{-1}(\xi) \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

and henceforth (12) follows from inequalities (a)–(d) in Lemma 7 below. The theorem is proved.

Lemma 7 ([H2]). *There exist positive constants c_j , $j = \overline{1, 4}$ such that for all $\xi \in \mathbb{C}$ with $\operatorname{Re} \xi > 0$ the following inequalities are satisfied:*

- (a) $\|\xi \tilde{L}^{-1}(\xi)\|_{\mathfrak{B}(\mathcal{H}_0, \mathcal{H}_0)} \leq c_1/|\xi|;$
- (b) $\|\tilde{L}^{-1}(\xi)\|_{\mathfrak{B}(\mathcal{H}_0, \mathcal{H}_{1/2})} \leq c_2/|\xi|;$
- (c) $\|\tilde{L}^{-1}(\xi) \tilde{C}\|_{\mathfrak{B}(\mathcal{H}_{1/2}, \mathcal{H}_0)} \leq c_3/|\xi|;$
- (d) $\|\tilde{L}^{-1}(\xi)(\tilde{B} + \xi I)\|_{\mathfrak{B}(\mathcal{H}_{1/2}, \mathcal{H}_{1/2})} \leq c_4/|\xi|.$

3.3. Solvability of the Cauchy problem. Due to theorem 6 we can easily study the generalized solutions to Cauchy problem (4)–(5). Let \mathbb{P} denote the orthoprojector in \mathbb{H} onto the first coordinate, i.e. $\mathbb{P}(x_1, x_2) = x_1$.

Theorem 8. *For any initial data $\psi_0 \in \mathcal{H}_{1/2}$, $\psi_1 \in \mathcal{H}$ Cauchy problem (4)–(5) has a unique generalized solution $u(t)$ such that*

$$(u(t), \dot{u}(t)) \rightarrow (\psi_0, \psi_1)$$

as $t \rightarrow 0$ in the norm of the space \mathbb{H} . This solution equals

$$u(t) = e^{k_0 t} \mathbb{P} \mathbb{U}_t \boldsymbol{\psi},$$

where $\boldsymbol{\psi} := (\psi_0, -k_0 \psi_0 + \psi_1)$, and satisfies the inequality

$$\|\dot{u}(t)\|_{\mathcal{H}}^2 + (\tilde{C}u(t), u(t)) \leq e^{k_0 t} (\|\psi_1\|_{\mathcal{H}}^2 + (\tilde{C}\psi_0, \psi_0)).$$

It is natural that in order to get a classical solution we should choose “smoother” initial data. Indeed, let $v(t)$ be a classical solution to problem (8)–(9) and $v(0) = \psi_0 \in \mathcal{H}_1$, $\dot{v}(0) = -k_0 \psi_0 + \psi_1 \in \mathcal{H}_{1/2}$. Applying to (8) the Laplace transform and integrating by parts, we get

$$\begin{aligned} 0 &= \int_0^\infty e^{-\xi t} (\ddot{v}(t) + \tilde{B}\dot{v}(t) + \tilde{C}v(t)) dt = \\ &= \tilde{L}(\xi) \int_0^\infty e^{-\xi t} v(t) dt - ((\tilde{B} + \xi I)\psi_0 - k_0 \psi_0 + \psi_1), \quad \operatorname{Re} \xi > 0 \end{aligned}$$

Applying now the inverse Laplace transform (see [V]) we arrive at the equality

$$(14) \quad v(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{\xi t} \tilde{L}^{-1}(\xi) [(\tilde{B} + \xi I)\psi_0 - k_0 \psi_0 + \psi_1] d\xi, \quad \sigma_0 > \xi_0.$$

Our aim is to prove that the function $v(t)$ defined by (14) coincides with $\mathbb{P} \mathbb{U}_t \boldsymbol{\psi}$ and is a classical solution to problem (8)–(9) (and hence the function $u(t) := e^{k_0 t} v(t)$ is a classical solution to Cauchy problem (4)–(5)).

Theorem 9. *Let $\psi_0 \in \mathcal{H}_1$ and $\psi_1 \in \mathcal{H}_{1/2}$; then the function $u(t) := e^{k_0 t} \mathbb{P} \mathbb{U}_t \boldsymbol{\psi}$ is a classical solution to Cauchy problem (4)–(5).*

Proof. First, the inequalities from Lemma 7 justify the possibility to apply the inverse Laplace transform in the form (14), as well as inclusions $v(t) \in \mathfrak{D}(\tilde{C}) = \mathcal{H}_1$ and $\dot{v}(t) \in \mathfrak{D}(\tilde{B})$. It remains to prove that the functions $v(t)$ and $\mathbb{P} \mathbb{U}_t \boldsymbol{\psi}$ coincide.

Notice that the holomorphic C_0 -semigroup \mathbb{U}_t can be constructed via its generator $\tilde{\mathbb{T}} \in \mathcal{A}(0, \varphi)$ by means of the integral (see [K])

$$\mathbb{U}_t \phi = \frac{1}{2\pi i} \int_{\gamma} e^{\xi t} (\tilde{\mathbb{T}} - \xi \mathbb{I})^{-1} \psi d\xi,$$

where the contour γ surrounds the sector $S(0, \varphi) := \{\xi \in \mathbb{C} \mid |\arg \xi - \pi| \leq \varphi\}$ (which contains the spectrum $\sigma(\tilde{\mathbb{T}})$ of the operator $\tilde{\mathbb{T}}$), and the integral converges strongly. Therefore, due to equality (13) we have

$$\mathbb{P}\mathbb{U}_t \psi = \frac{1}{2\pi i} \int_{\gamma} e^{\xi t} \tilde{L}^{-1}(\xi) [(\tilde{B} + \xi I)\psi_0 - k_0\psi_0 + \psi_1] d\xi,$$

which coincides with (14) as the integrand is an analytical function and hence the contour γ can be transformed into the one from (14). The theorem is proved.

4. APPLICATION TO PROBLEM (1)–(2)

If the operators A , B , and G are originated by system (1)–(2) and so are defined by (6), we can essentially refine many of the above-listed results. Let $g_{\pm}(x) := \frac{1}{2}(|g(x)| \pm g(x))$.

Lemma 10. (cf. Lemma 2) (a) *If $g_{-}(x) \equiv 0$, then all the nonreal EV's of the pencil $L(\lambda)$ belong to the disc*

$$D := \{\lambda \in \mathbb{C} \mid |\lambda + 1/\alpha| \leq 1/\alpha\}.$$

(b) *If $g_{+}(x) \equiv 0$, then the nonreal spectrum of the pencil $L(\lambda)$ lies outside of the disc*

$$D' := \{\lambda \in \mathbb{C} \mid |\lambda + 1/\alpha| < \sqrt{1 - \alpha b_{+}}/\alpha\},$$

where $b_{+} := \text{esssup } \beta(x)$.

According to Lemma 3, the number $\varkappa_1(L)$ of (real) EV's in the right half-plane equals $\nu(T)$, the total multiplicity of negative spectrum of the operator $T := A + G = \frac{d^4}{dx^4} + \frac{d}{dx}(x)\frac{d}{dx}$ with the domain $\mathfrak{D}(T) = \{y \in W_2^4(\mathbb{R}_{+}) \mid y(0) = y'(0) = 0\}$. Consider also the Schrödinger operator $S := -\frac{d^2}{dx^2} - g(x)$ with the domain $\mathfrak{D}(S) = \{y \in W_2^2(\mathbb{R}_{+}) \mid y(0) = 0\}$, and by $\nu(S)$ denote the (possibly infinite) number of its negative EV's. The crucial role in estimating of $\nu(T)$ plays the following statement.

Lemma 11 ([H1]). *The following inequality holds:*

$$\nu(T) \leq \nu(S) \leq \nu(T) + 1.$$

Corollary 12. (a) *Suppose that $g(x) \geq 0$ for all sufficiently large x and*

$$\sup_{x \geq 0} t \int_t^{\infty} g_{+}(s) ds = \infty$$

Then $\nu(T) = \infty$, and $\lambda = 0$ is an accumulation point of real EV's from both sides.

(b) If $\max_{x \geq a} x \int_x^\infty g_+(s) ds \leq 1/4$ for some $a > 0$, then $\nu(T) < \infty$, and EV's do not accumulate at the point 0 from the right. In particular, we then have

$$\nu(T) \leq \int_0^\infty x g_+(x) dx.$$

Proof. Analogous statements for the Schrödinger operator S are well-known, see, e.g., [B] and [RS].

What concerns accumulation of the real EV's at the point $\lambda = -1/\alpha$, we have the following result.

Lemma 13. *Suppose that $g_+(x) \not\equiv 0$; then $\lambda = -1/\alpha$ is an accumulation point of real EV's of the pencil $L(\lambda)$ from the right. If $b_+ < 1/\alpha$, then this condition is a necessary one for accumulation.*

Finally, we can describe the properties of solution to problem (1)–(3) in terms of initial data.

Theorem 14. *Suppose that*

$$\psi_0(x) \in W_{2,U}^2(\mathbb{R}_+) := \{y(x) \in W_2^2(\mathbb{R}_+) \mid y(0) = y'(0) = 0\}$$

and $\psi_1(x) \in L_2(\mathbb{R}_+)$. Then problem (1)–(3) has a unique generalized solution $u(x, t)$. If, in addition, $\psi_0(x)$ belongs to $W_2^4(\mathbb{R}_+)$, and $\psi_1(x)$ belongs to $W_{2,U}^2(\mathbb{R})$, then the solution $u(x, t)$ is a classical one.

Acknowledgements. The author is deeply grateful to Prof. A. A. Shkalikov for very useful and stimulating discussions.

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